Limit theorem for the high-frequency asymptotics of the multivariate Brownian semistationary process

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Joint Work with Dr. Almut Veraart

3rd Young Researchers Meeting in Probability, Numerics and Finance
Le Mans, 01 July 2016

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Outline

The Brownian Semistationary Process

Definition

The one-dimensional Brownian semistationary process ($\mathcal{BSS}$) is defined as:

$$Y_t = \int_{-\infty}^{t} g(t-s)\,\sigma_s dW_s, \qquad (1)$$

where W is an $\mathscr{F}_t$-adapted Brownian measure, σ is càdlàg and $\mathscr{F}_t$-adapted, $g\colon \mathbb{R} \to \mathbb{R}$ is a deterministic function, continuous in $\mathbb{R} \setminus \{0\}$, with $g(t) = 0$ if $t \leq 0$ and $g \in L^2((0,\infty))$. We also need to impose that $\int_{-\infty}^{t} g^2(t-s)\sigma_s^2\, ds < \infty$ a.s. so that a.s. we have $Y_t < \infty$ for all $t \geq 0$.

Basic properties

1. For $\sigma \equiv 1$, the Gaussian core

$$G_t := \int_{-\infty}^{t} g(t-s)\, dW_s,$$

is Gaussian, with mean 0 and variance $\int_0^\infty g^2(s)\, ds$.

2. The process is second order stationary if σ is.
3. It does not have independent increments.
4. It is a typical assumption that $g(x) \sim x^\delta$ around 0. By Kolmogorov-Centsov, then the process has a modification with α-Hölder continuous sample paths, for all $\alpha \in (0, \delta + \frac{1}{2})$.

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Semimartingale issues

Let us look again at the simple case where $\sigma = 1$:

$$G_t = \int_{-\infty}^{t} g(t-s)\, dW_s.$$

Then we can write a small increment as:

$$G_{t+dt} - G_t = \int_{-\infty}^{t+dt} g(t+dt-s)\, dW_s - \int_{-\infty}^{t} g(t-s)\, dW_s.$$

Adding and subtracting the same quantity:

$$G_{t+dt} - G_t = \int_{-\infty}^{t+dt} (g(t+dt-s) - g(t-s))\, dW_s + \int_{t}^{t+dt} g(t-s)\, dW_s.$$

Letting $dt \to 0$, we (heuristically) get:

$$dG_t = \int_{-\infty}^{t} g'(t-s)\, dW_s + g(0+) dW_t.$$

We see that we have a problem if $g' \notin L^2(\mathbb{R})$, or $g(0+) = \infty$.

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Why the BSS process?

1. The Brownian semistationary process has been used in the context of turbulence modelling, as a model for the field of the velocity vectors in a turbulent flow. Then $g(x) \sim x^{-\frac{1}{6}}$ fits well with Kolmogorov's scaling law.
2. In finance, the BSS process has successfully been used in the modelling of energy prices.
   Arbitrage?!
3. It is possible to ensure that no arbitrage holds even if non semimartingales are used as price processes, provided that they satisfy conditions that ensure existence of the so-called consistent price systems: i.e. the existence of a semimartingale that evolves within the bid-ask spread, for which there exists an equivalent martingale measure. (Jouini and Kallal)

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Consistent Price System

Under arbitrarily small transaction costs, Guasoni, Rásonyi and Schachermayer, showed that a price process X_t has a Conditional price system if it has the so-called conditional full support property:

$$\text{Supp}\left(\text{Law}\{X_u | t \leq u \leq T | \mathscr{F}_t\}\right) = C_{X_t}[t; T]$$

Fractional Brownian motion, which can be expressed as:

$$X_t = \int_{-\infty}^{t} \left(f(s-t) - f(s)\right) \, dB_s$$

has this property. (Cherny)
Pakkanen, finally, finds that our BSS process with stochastic volatility possesses this property too.

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Quadratic variation

Let the increments of Y be denoted by $\Delta_i^n Y := Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$. Outside the semimartingale class, we do not have any guarantee that

$$[Y]_t := \mathbb{P} - \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta_i^n Y)^2$$

exists.

Indeed, take for example the fractional Brownian motion B^H. Then one can show that in L^2:

- $$\sum_{i=1}^{n} (\Delta_i^n B^H)^2 \to +\infty \qquad \text{if } H < \frac{1}{2}$$
- $$\sum_{i=1}^{n} (\Delta_i^n B^H)^2 \to 0 \qquad \text{if } H > \frac{1}{2}$$

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Limit theorem setting

We want to study convergence of the realised variation process. We work in a finite horizon $[0, T]$. Fix a number $n \in \mathbb{N}$ and let $\Delta_i^n Y := Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$. Consider the process:

$$X_t^{(n)} := \sum_{i=1}^{\lfloor nt \rfloor} (\Delta_i^n Y)^2 ,$$

or, more generally,

$$X_t^{(n)} := \sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)}.$$

If we let $n \to \infty$, what kind of convergence can we get? In probability, in distribution? Can we get a Donsker-type result? (Note that for each n, $X^{(n)}$ has discontinuous paths.).

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Convergence of processes

Definition (u.c.p. convergence)

The sequence of càdlàg processes $X^{(n)}$ is said to converge *uniformly on compacts in probability* (u.c.p.) to X if, for all $t \leq T$ and all $\varepsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in[0,t]} \left| X_s^{(n)} - X_s \right| > \varepsilon \right) = 0$$

Theorem

Suppose that, for all t in a dense subset $D \subset [0, T]$, $X_t^{(n)} \xrightarrow{\mathbb{P}} X_t$. Assume further, that the paths of $X^{(n)}$ are increasing with time and the paths of X are continuous, almost surely. Then, the (stronger) convergence

$$X_\cdot^{(n)} \xrightarrow{u.c.p.} X_\cdot$$

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Stable convergence

Definition

Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be fixed. Suppose the sequence of variables $Y^{(n)}$ converges weakly to Y, denoted:

$$Y^{(n)} \Rightarrow Y.$$

We say that $Y^{(n)}$ converges *stably* to Y if, for any $\mathcal{F}-$measurable set B, we have:

$$\lim_{n\to\infty} \mathbb{P}\left(\{Y^{(n)} \leq x\} \cap B\right) = \mathbb{P}\left(\{Y \leq x\} \cap B\right),$$

for a countable, dense set of points x.

Equivalently, if, for any f bounded Borel function, and for any $\mathcal{F}-$measurable *fixed* variable Z:

$$\lim_{n\to\infty} \mathbb{E}\left[f\left(Y^{(n)}\right) Z\right] = \mathbb{E}\left[f(Y)Z\right]$$

Equivalently, $(Y^{(n)}, Z) \Rightarrow (Y, Z)$.

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(Counter)example

Unlike convergence in distribution, stable convergence in distribution is a property of the sequence of rv's $Y^{(n)}$ rather than of the corresponding sequence of distribution functions. Take X and $\tilde{X}$ be independent with a common distribution. Set

$$Z^{(n)} = \begin{cases} X & \text{if } n \text{ is odd} \\ \tilde{X} & \text{if } n \text{ is even} \end{cases} .$$

Obviously, $Z^{(n)} \Rightarrow X$, but the convergence is not stable. Take for example $B = \{X \leq x\}$:

$$\mathbb{P}\left(\{Z^{(n)} \leq x\} \cap B\right) = \begin{cases} F_X(x) & \text{if } n \text{ is odd} \\ F_X^2(x) & \text{if } n \text{ is even} \end{cases}$$

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Multivariate setting

Take $W^{(1)}$ and $\tilde{W}$ two independent Brownian measures and consider a continuous stochastic process $(\rho_t)_{t\in\mathbb{R}}$ defined on the whole real line.

Definition (Two-dimensional $\mathcal{BSS}$ without stochastic volatility)

$$Y_t^{(1)} := \int_{-\infty}^{t} g^{(1)}(t-s)\sigma_s^{(1)}\, dW_s^{(1)}$$

$$Y_t^{(2)} := \int_{-\infty}^{t} g^{(2)}(t-s)\sigma_s^{(2)}\rho_s\, dW_s^{(1)} + \int_{-\infty}^{t} g^{(2)}(t-s)\sigma_s^{(2)}\sqrt{1-\rho_s^2}\, d\tilde{W}_s.$$

The vector process: $(\mathbf{Y}_t)_{t\in\mathbb{R}}$ is defined to be a 2-dimensional correlated Brownian semistationary process.

Assumption

ρ has continuous sample paths, is independent of $W^{(1)}$ and $\tilde{W}$, and its paths lie in the interval $[-1,+1]$.

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Law of large numbers

The first result we want to prove is a law of large numbers for the realised covariation.

$$\frac{1}{n}\frac{\sum_{i=1}^{\lfloor n\cdot\rfloor}\Delta_i^n Y^{(1)}\Delta_i^n Y^{(2)}}{c(\Delta_n)} \overset{u.c.p.}{\to} \int_0^{\cdot} \sigma_s^{(1)}\sigma_s^{(2)}\rho_s\, ds,$$

for a certain scaling factor $c(\Delta_n)$. (Δ_n is short for $\frac{1}{n}$).

Assumption

We require that, for $i \in \{1,2\}$, the quantities:

$$\int_0^x \left(g^{(i)}(s)\right)\left(g^{(j)}(s)\right)\, ds \tag{2}$$

$$\int_0^1 \left(g^{(i)}(s+x) - g^{(i)}(s)\right)\left(g^{(j)}(s+x) - g^{(j)}(s)\right)\, ds \tag{3}$$

can be written as $x^{2\delta^{(i)}+1}L^{(i,j)}(x)$, for $x \to 0+$, for $\delta^{(i)} \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, and $L^{(i,j)}$ a slowly varying function.

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Ideas of proof

We consider the sigma algebra $\mathscr{H} := \mathscr{F}^{\rho,\sigma^{(1)},\sigma^{(2)}}$ generated by the processes $\rho, \sigma^{(1)}, \sigma^{(2)}$. We perform the splitting:

$$\left| \frac{1}{n} \frac{\sum_{i=1}^{n} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)}}{c(\Delta_n)} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\frac{1}{c(\Delta_n)} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \middle| \mathscr{H} \right] \right| + \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\frac{1}{c(\Delta_n)} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \middle| \mathscr{H} \right] - \int_0^1 \sigma_l^{(1)} \sigma_l^{(2)} \rho_l \, dl \right|. \quad (4)$$

If we compute:

$$\mathbb{E}\left[\Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \middle| \mathscr{H} \right] = \int_0^\infty \varphi_{\Delta_n}^{(1)} \varphi_{\Delta_n}^{(2)} \sigma_{i\Delta_n - s}^{(1)} \sigma_{i\Delta_n - s}^{(2)} \rho_{i\Delta_n - s} \, ds,$$

where

$$\varphi_{\Delta_n}^{(i)}(s) = \begin{cases} g^{(i)}(s) & s \leq \Delta_n \\ g^{(i)}(s) - g^{(i)}(s - \Delta_n) & s > \Delta_n \end{cases}$$

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If we compute:

$$\mathbb{E}\left[\Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \middle| \mathscr{H} \right] = \int_0^\infty \varphi_{\Delta_n}^{(1)} \varphi_{\Delta_n}^{(2)} \sigma_{i\Delta_n - s}^{(1)} \sigma_{i\Delta_n - s}^{(2)} \rho_{i\Delta_n - s} \, ds,$$

where

$$\varphi_{\Delta_n}^{(i)}(s) = \begin{cases} g^{(i)}(s) & s \leq \Delta_n \\ g^{(i)}(s) - g^{(i)}(s - \Delta_n) & s > \Delta_n \end{cases}$$

So we can see:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{c(\Delta_n)}\mathbb{E}\left[\Delta_i^n Y^{(1)}\Delta_i^n Y^{(2)}\middle|\mathscr{H}\right] =$$

$$\frac{\int_0^\infty \varphi^{(1)}_{\Delta_n}(s)\varphi^{(2)}_{\Delta_n}(s)\frac{1}{n}\left(\sum_{i=1}^n \sigma^{(1)}_{i\Delta_n-s}\sigma^{(2)}_{i\Delta_n-s}\rho_{i\Delta_n-s}\right)\,ds}{c(\Delta_n)} =$$

$$\int_{\mathbb{R}^+}\frac{1}{n}\left(\sum_{i=1}^n \sigma^{(1)}_{i\Delta_n-s}\sigma^{(2)}_{i\Delta_n-s}\rho_{i\Delta_n-s}\right)\,d\pi_n(s) \quad (5)$$

$$\frac{d\pi_n}{ds} = \frac{\varphi^{(1)}_{\Delta_n}(s)\varphi^{(2)}_{\Delta_n}(s)}{c(\Delta_n)}$$

So in order for π_n to be a probability measure, we need to ask that

$$c(\Delta_n) = \int_0^\infty \varphi^{(1)}_{\Delta_n}(s)\varphi^{(2)}_{\Delta_n}(s)\,ds.$$

Now if $\pi_n \Rightarrow \pi$, then we have the almost sure convergence:

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We have the limit:

$$\int_{\mathbb{R}^+} \left(\int_{-s}^{1-s} \rho_l \sigma_l^{(1)} \sigma_l^{(2)} \, dl \right) \, d\pi(s).$$

If we can show that actually: $\pi = \delta_0$, the limit becomes:

$$\int_0^1 \rho_l \sigma_l^{(1)} \sigma_l^{(2)} \, dl.$$

Theorem

If there exist β such that $\left((g^{(1)}(x))'\right)^2$ and $\left((g^{(2)}(x))'\right)^2$ are non increasing for $x > \beta$, then:

$$\pi_n \Rightarrow \delta_0$$

Example

For example, the Gamma kernel:

$$g(x) = x^{\delta} e^{-\lambda x}$$

satisfies this condition for $\delta \in (-\frac{1}{2}, 0)$.

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Central Limit Theorem

Consider a bivariate Gaussian process:

$$\mathbf{G}_t = \begin{pmatrix} G_t^{(1)} \\ G_t^{(2)} \end{pmatrix} = \begin{pmatrix} \int_{-\infty}^t g^{(1)}(t-s)\, dW_s^{(1)} \\ \int_{-\infty}^t g^{(2)}(t-s)\, dW_s^{(2)} \end{pmatrix}$$

with $dW^{(1)} dW^{(2)} = \rho\, dt$ for a constant ρ. Let H be the Hilbert space generated by the standard Gaussian random variables:

$$\left(\frac{\Delta_j^n G^{(h)}}{\tau_n^{(h)}} \right)_{n \geq 1, 1 \leq j \leq \lfloor nt \rfloor, h \in \{1,2\}} .$$

with the scalar product induced by their covariance.

We will assume the existence of an isometry $B \colon \mathcal{H} \to H$ between a separable Hilbert space $\mathcal{H}$ and H, such that:

$$\mathbb{E}\left[B(h_1)B(h_2)\right] = \langle h_1, h_2 \rangle_{\mathcal{H}}.$$

B is called an isonormal Gaussian process.

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Tiny, tiny bits of Malliavin calculus

A fundamental result in Malliavin calculus is the Wiener-Itô chaos decomposition:

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the linear space generated by the variables $H_n(B(h))$ and H_n is the n-th Hermite polynomial. $\mathcal{H}_n$ is called the n-th Wiener chaos.
There exists an isometry:

$$I_p \colon H^{\odot p} \to \mathcal{H}_p \subset L^2(\Omega)$$

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We can write:

$$\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} = I_1\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\right) I_1\left(\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right),$$

from which:

$$\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} = I_2\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \widetilde{\otimes} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right) + \mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right].$$

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As customary, to prove weak convergence, we need two ingredients:

1. Tightness
2. Convergence of the finite dimensional distributions.

$$I_2(f_{k,n})=I_2\left(\frac{1}{\sqrt{n}}\sum_{i=\lfloor na_k\rfloor+1}^{\lfloor nb_k\rfloor}\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\tilde{\otimes}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right)$$

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Convergence within fixed Wiener chaos

Theorem

Let $d \geq 2$ and $q_d, \ldots, q_1 \geq 1$ be some fixed integers. Consider vectors:

$$\mathbf{F}_n := (F_{1,n}, \ldots, F_{d,n}) = (I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})), \qquad n \geq 1,$$

with $f_{i,n} \in H^{\odot q_i}$. Let $C \in \mathscr{M}_d(\mathbb{R})$ be a symmetric, non-negative definite matrix, and let $\mathbf{N} \sim \mathscr{N}_d(0, C)$. Assume that:

$$\lim_{n\to\infty} \mathbb{E}\left[F_{r,n}F_{s,n}\right] = C(r,s), \qquad 1 \leq r, s \leq d. \tag{7}$$

Then, as $n \to \infty$ the following two conditions are equivalent:

a) $\mathbf{F}_n$ *converges in law to* $\mathbf{N}$.

b) *For every $1 \leq r \leq d$, $F_{r,n}$ converges in law to $\mathscr{N}(0, C(r,r))$.*

The Fourth Moment Theorem

The Gaussian distribution is identified by its moments. That is,

$$X \sim N(0,1) \qquad \textcolor{red}{\text{if and only if}} \qquad \mathbb{E}[X^n] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n!! & \text{if } n \text{ is even.} \end{cases}$$

Theorem (Nualart and Peccati)

Let $F_n = I_q(f_n), n \geq 1$, be a sequence of random variables belonging to the q-th chaos of X, for some fixed integer $q \geq 2$ (so that $f_n \in H^{\odot q}$). Assume, moreover, that $\mathbb{E}[F_n^2] \to \sigma^2 > 0$ as $n \to \infty$. Then, as $n \to \infty$, the following assertions are equivalent:

1. $F_n \xrightarrow{\mathscr{L}} N(0, \sigma^2)$,
2. $\lim_{n\to\infty} \mathbb{E}[F_n^4] = 3\sigma^2$,
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Assumption

1. $\mathbb{E}\left[G^{(j)}_{s+t}G^{(i)}_s\right] = \int_0^{+\infty} g^{(i)}(s)g^{(j)}(s+t)\rho_{i,j}\,ds = t^{\beta^{(i)}+\beta^{(j)}-1}L_0^{(i,j)}(t)$
2. $\mathbb{E}\left[\left(G^{(i)}_{t+k} - G^{(i)}_k\right)^2\right] = t^{2\beta^{(i)}-1}L_0^{(i)}(t) \Rightarrow \sqrt{R^{(i)}(t)R^{(j)}(t)} = t^{\beta^{(i)}+\beta^{(j)}-1}\tilde{L}_0(t)$
3. $\mathbb{E}\left[\left(G^{(i)}_{t+k} - G^{(i)}_k\right)^2\right]'' = t^{\beta^{(i)}+\beta^{(j)}-3}\tilde{L}_2^{(i,j)}(t)$
4. $\limsup_{x\to 0^+} \sup_{y\in[x,x^b]} \left|\frac{L_2^{(i,j)}(y)}{\tilde{L}_0(x)}\right| < \infty$

Theorem (Weak Convergence of the Gaussian Core)

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}}\frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}\right]\right)\right)_{t\in[0,T]} \overset{st.}{\Rightarrow} \left(\sqrt{\beta}B_t\right)_{t\in[0,T]},$$

where B_t is a Brownian motion independent of the processes $G^{(1)}$, $G^{(2)}$, β is the limiting standard deviation and the convergence is in the Skorokhod space $\mathcal{D}[0,T]$ equipped with the Skorokhod topology.

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For Further Reading I

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